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The Dirac oscillator of arbitrary spin

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Abstract. The problem of a relativistic particle with an arbitrary spin has had an old and distinguished history, particularly if the particle is free or interacting with an electromagnetic field. Recently a new type of interaction, denoted as the Dirac oscillator, has been introduced, first for a particle of spin $\frac{1}{2}$ and later for spins 0 and 1, and it has suggested a possible generalization to arbitrary spin. Following a procedure originally developed for a system of n particles with spin $\frac{1}{2}$ with a Dirac oscillator interaction, we have derived a *single* particle equation with spins in the range $\frac{1}{2}n, \frac{1}{2}n-1, \ldots, \frac{1}{2}$ or 0 and through a symmetric representation of the permutation group restricted it to spin $\frac{1}{2}n$. We have solved the resulting equation explicitly by reducing it to an algebraic one in the energy E, whose coefficients depend on the number of quanta N, the total angular momentum j and the frequency ω of the oscillator. Properties of the energy spectra will be discussed for spins $\frac{1}{2}$, 1, $\frac{3}{2}$.

1. Introduction

The problem of a relativistic particle with arbitrary spin has had a long and distinguished history, particularly when the particle is free or interacting with an electromagnetic field [1–4], although even at the present date open questions remain in relation to this problem.

Recently a renewed interest in the subject appeared when it was shown that relativistic particles of spin $\frac{1}{2}$ could be acted on by a simple type of potential to which Moshinsky and Szczepaniak [5] gave the name of Dirac oscillator. The reasoning leading to this interaction parallelled that of Dirac that linearized the quadratic Klein–Gordon equation. If an oscillator interaction, quadratic in the coordinates, is added to the latter it suggests that linearization could be achieved if the momentum $\boldsymbol{p}$ in the free particle Dirac equation is replaced by some linear combination of $\boldsymbol{p}$ and $\boldsymbol{r}$, where the latter is the position vector. In this way one arrives at the equation [5]

$$\mathrm{i}(\partial\psi/\partial t) = [\boldsymbol{\alpha}\cdot(\boldsymbol{p} - \mathrm{i}\omega\boldsymbol{r}\beta) + \beta]\psi \tag{1.1}$$

in which the units used are those where

$$\hbar = m = c = 1 \tag{1.2}$$

with m being the mass of the particle and c the velocity of light and $\boldsymbol{\alpha}, \beta$ are the 4×4 matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \tag{1.3}$$

with $\boldsymbol{\sigma}$ being the vector whose components are the Pauli spin matrices while ω is the frequency of the oscillator in the units mentioned in equation (1.2).

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Actually, an equation of the form (1.1) had been considered before [6] its appearance in [5], but it did not lead to further applications, while the Dirac oscillator [5] is now referred in an extensive literature, both in its one- [7] and many-body versions [8].

While equation (1.1) does not look explicitly Lorentz invariant, it can be shown to have this property with the help of a unit timelike four vector as indicated, for example, in [9].

Thus a relativistic equation for a particle of spin $\frac{1}{2}$ with a Dirac oscillator interaction became available, but it raised the question of whether similar equations could be obtained for spins different from $\frac{1}{2}$. One was aware, from the work of Kemmer [3] and others, that relativistic equations linear in the four momentum of the particle were available for spin 0 and 1. Using this fact and the reasoning that led to the Dirac oscillator Nedjadi and Barret [10] arrived at relativistic oscillator equations of spin 0 and 1.

The availability of what we shall continue to call Dirac oscillators, but now of spin 0 and 1, besides $\frac{1}{2}$, immediately raises the question of whether the concept can be generalized to arbitrary spin. The free-particle relativistic equations of arbitrary spin have been proposed but they either involve bothersome constraints or start from as many Dirac equations as that which are required to get the desired spin from its original $\frac{1}{2}$ value [4]. Thus we decided to follow a different procedure based on our use of Barut's approach [11] for a single relativistic equation for the many-body problem [12]. Instead of many particles we shall consider in the next section the approach mentioned for a *single* particle but with arbitrary spin, and later introduce into the equation the Dirac oscillator interaction. Our results will be discussed in a general way but then analysed in detail for spins $\frac{1}{2}$, 1, $\frac{3}{2}$.

2. Relativistic equations for arbitrary spin

It is well known that if in a non-relativistic problem we have several states of spin $\frac{1}{2}$ we can, by direct products, get a state of higher spin. As the Dirac equation corresponds to a relativistic particle of spin $\frac{1}{2}$ and in it there appear matrix components γ^μ, $\mu = 0, 1, 2, 3$, of a four vector, related to the α_i, β of (1.3) by

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha_i \qquad i = 1, 2, 3 \tag{2.1}$$

we can expect that by direct products of these γ_s^μ with different indexes $s = 1, 2, \ldots, n$, i.e.

$$\gamma_s^\mu = \mathbf{I} \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} \otimes \gamma^\mu \otimes \mathbf{I} \otimes \cdots \otimes \mathbf{I} \tag{2.2}$$

we could obtain relativistic equations of higher spin. Note that in (2.2) we have direct products of 4×4 unit matrices $\mathbf{I}$ and in the position s the γ^μ. Also the metric tensor we shall use in the present paper will be of the form

$$g_{\mu\nu} = 0 \qquad \mu \neq \nu \qquad g_{11} = g_{22} = g_{33} = -g_{00} = 1. \tag{2.3}$$

How do we combine the γ_s^μ so that we get a single relativistic equation that contains an arbitrary spin and represents either a free particle or one in a Dirac oscillator?

For the purpose indicated in the previous paragraph it is convenient to recall Barut's procedure [11] for obtaining single many-body relativistic equations from field theory, which inspired the approach of Moshinsky *et al* [12] which starts with a Lorentz invariant formulation of the free-particle many-body problem. It is convenient in the approaches mentioned to introduce first a timelike unit four vector u_μ, $\mu = 0, 1, 2, 3$, which implies that there is a frame of reference in which it can take the value

$$(u_\mu) = (1, 0, 0, 0). \tag{2.4}$$

With the help of this four vector and the γ_s^μ of (2.2) we can define the Lorentz scalars

$$\Gamma = \prod_{r=1}^{n} (\gamma_r^\mu u_\mu) \qquad \Gamma_s = (\gamma_s^\mu u_\mu)^{-1}\Gamma \tag{2.5}$$

where repeated Greek indexes μ are summed over their values $\mu = 0, 1, 2, 3$. Note that $(\gamma_s^\mu u_\mu)^{-1}$ in Γ_s just eliminates the corresponding term in Γ and thus Γ_s is still in product form.

In [12] we showed that with the help of the Lorentz scalar operators Γ_s and the γ_s^μ, as well as that of the four momenta of the n particles, it was possible to get a *single* Lorentz invariant equation for the non-interacting n-body system, and later we extended it to include Dirac oscillator interactions. In the present problem we wish to consider only one particle and thus we have the single four momentum p_μ, $\mu = 0, 1, 2, 3$. However, we want to have a formalism in which spin acquires arbitrary values, and this suggests we use equation (3.29) of [12] but take all the momenta equal which leads, in our units, to the Lorentz invariant equation

$$\sum_{s=1}^{n} \Gamma_s (\gamma_s^\mu p_\mu + 1)\psi = 0. \tag{2.6}$$

It is useful noting, as in [12], that for an equation (2.6) corresponding to a many-body problem where p_μ is replaced by $p_{\mu s}$, $s = 1, 2, \ldots, n$, the unit timelike vector (u_μ) has a clear physical and conserved meaning as it becomes the normalized four momentum of the whole system. As this interpretation is not valid for the single particle of equation (2.6), it would seem at first sight to make the appearance there of (u_μ) a completely formal device. This is not though the case because, while we shall consider in the analysis given below a (u_μ) that has the value $(1, 0, 0, 0)$, its presence would allow us to write the solution of equation (2.6) also in any other frame of reference by a boost associated with the components of (u_μ) in this new frame [9].

At this point it is worthwhile remarking that as long ago as 1939 Kemmer [3] noted that, except for the factor Γ_s and for $n = 2$, an equation similar to the above, (in equations (54) and (55) of [3]) could represent particles of both spin 0 and 1. It would thus be reducible with respect to spin but no longer restricted to spin $\frac{1}{2}$. We shall show later that we can limit the solution of equation (2.6) to a definite spin by making use of its invariance with respect to permutation of the index s, and restricting ourselves to a definite irrep of the later group.

Before analysing equation (2.6) in the frame of reference in which u_μ takes the form (2.4), it is convenient to consider, in the Lorentz invariant form, the generalization of the former equation to include a Dirac oscillator interaction. For this purpose we first define the transverse part of the position four vector x_μ as

$$x_{\perp\mu} \equiv x_\mu + (x_\nu u^\nu) u_\mu \tag{2.7}$$

which of course will have the property that when u_μ takes the form (2.4), and using the metric (2.3), $x_{\perp 0} = 0$ and only the spatial part of the vector $x_{\perp i}$, $i = 1, 2, 3$, remains.

If in equation (2.6) we replace p_μ by $p_\mu - \mathrm{i}\omega x_{\perp\mu}\Gamma$, which is similar to what is done in equation (3.36) of [12], as well as for the spatial part in equation (1.1) of the present paper, we then get the Lorentz invariant expression

$$\sum_{s=1}^{n} \Gamma_s [\gamma_s^\mu (p_\mu - \mathrm{i}\omega x_{\perp\mu}\Gamma) + 1]\psi = 0 \tag{2.8}$$

which will be our fundamental equation for discussing the Dirac oscillator with arbitrary spin.

We now wish to write equation (2.8) in the frame of reference in which u_μ takes the value $(1, 0, 0, 0)$ because, as shown in [12], it is the simplest one in which we can carry out our analysis. Clearly equation (2.8) then takes the form

$$\left\{n\Gamma^0 p_0 + \sum_{s=1}^{n} \Gamma_s^0[\boldsymbol{\gamma}_s \cdot (\boldsymbol{p} - \mathrm{i}\omega \boldsymbol{r} B) + 1]\right\}\psi = 0 \tag{2.9}$$

with

$$\Gamma^0 = \prod_{r=1}^{n} \gamma_r^0 = \beta \otimes \beta \otimes \beta \otimes \cdots \otimes \beta \equiv B \qquad \Gamma_s^0 = (\gamma_s^0)^{-1}\Gamma^0 \tag{2.10}$$

and $\boldsymbol{\gamma}_s$, $\boldsymbol{p}$, $\boldsymbol{r}$ now being ordinary three component vectors. Multiplying by Γ^0 and making use of relations (2.1), as well as of the fact that the dependence on x^0 can be of the form $\exp(-\mathrm{i}Ex^0)$ with E being the energy, so that $p_0 \exp(-\mathrm{i}Ex^0) = -E \exp(-\mathrm{i}Ex^0)$, we finally arrive at the expression

$$\sum_{s=1}^{n} [\boldsymbol{\alpha}_s \cdot (\boldsymbol{p} - \mathrm{i}\omega \boldsymbol{r} B) + \beta_s]\psi = nE\psi \tag{2.11}$$

for the equation with Dirac oscillator interaction. In the case of the free particle we just have to take $\omega = 0$ in (2.11).

We note immediately that equation (2.11) is invariant under permutation of the index s of the $\boldsymbol{\alpha}_s$ and β_s matrices and this applies also to B which could be written as the product $B = \beta_1\beta_2 \ldots \beta_n$.

However, note that we should be careful about what we understand as permutation of the indexes. If we have, for example, $n = 3$ then

$$\boldsymbol{\alpha}_1 = \begin{pmatrix} 0 & \boldsymbol{\sigma}_1 \\ \boldsymbol{\sigma}_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{2.12}$$

and if we apply to it the transposition $[1, 3]$ we have to change the index of the $\boldsymbol{\sigma}$'s from 1 to 3, but also interchange the matrices in the direct product so that we get

$$[1, 3]\boldsymbol{\alpha}_1 = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \otimes \begin{pmatrix} 0 & \boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_3 & 0 \end{pmatrix} = \boldsymbol{\alpha}_3. \tag{2.13}$$

Under this type of operation it is clear that equation (2.11) remains invariant.

In the next section we shall proceed to discuss the solution of equation (2.11) so as to obtain the energy spectrum, assuming wavefunctions are symmetric under permutation of the states, which guarantees, as Bargmann and Wigner [4] showed, that we are dealing with states of spin $(n/2)$.

3. Eigenvalues and eigenfunctions for a Dirac oscillator with arbitrary spin

Starting from equation (2.11) we shall discuss the procedure of obtaining the eigenvalues of the energy as a function of the number of quanta N and the total angular momentum j, when the wavefunction is completely symmetric under interchanges of the type indicated in (2.13), where the latter restriction was considered in the work of Bargmann and Wigner [4].

We begin by taking a wavefunction in terms of indexes that indicate large and small components, and later rephrase the whole problem in terms of matrices.

3.1. The components $\psi_{\tau_1\tau_2\ldots\tau_n}$ of the wavefunction

When considering the problem of equation (1.1) where the spin was $\frac{1}{2}$ we proposed a solution [5]

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \exp(-\mathrm{i}Et) \tag{3.1}$$

where ψ_τ, $\tau = 1, 2$, depend only on the coordinates. We immediately arrived at a solution of the problem, as the effective Hamiltonian [5] turned out to be that of an oscillator with a strong spin–orbit term.

The above experience suggests that for arbitrary spin, i.e. in equation (2.11), the ψ should have n indexes τ instead of a single one and thus its components could be denoted by [13]

$$\psi_{\tau_1\tau_2\ldots\tau_n} \tag{3.2}$$

where all τ_s, $s = 1, 2, \ldots, n$, take the value 1 or 2. Note that the τ's are *not* spinor indexes but rather indicate, for example, for positive energy, the large or small character of the wavefunction [5] associated with the index s.

We now wish to see the effect of the operators B, β_s, $\boldsymbol{\alpha}_s$ appearing in (2.11) on the components of the wavefunction indicated in (3.2). It is clear from (2.10) that B when acting on an index τ_s, $s = 1, 2, \ldots, n$, gives 1 if $\tau_s = 1$ and -1 if $\tau_s = 2$, thus corresponding to the phase factor $(-1)^{1+\tau_s}$. For the set of all indexes we thus have

$$B\psi_{\tau_1\ldots\tau_n} = (-1)^{n+\tau_1+\tau_2+\cdots+\tau_n}\psi_{\tau_1\tau_2\ldots\tau_n}. \tag{3.3}$$

The effect of β_s (whose definition is as that of γ_s^μ of (2.2) when $\mu = 0$) is clearly given by

$$\beta_s\psi_{\tau_1\ldots\tau_s\ldots\tau_n} = (-1)^{1+\tau_s}\psi_{\tau_1\ldots\tau_s\ldots\tau_n}. \tag{3.4}$$

For $\boldsymbol{\alpha}_s$ whose expression is given in terms of a direct product similar to that of (2.2), we see that it acts only on the index τ_s, and if $\tau_s = 1$ (or 2) it transforms it into $\tau_s = 2$ (or 1) and at the same time applies the Pauli matrix operator $\boldsymbol{\sigma}_s$ to ψ. Introducing then the 2×2 matrix $\|\epsilon_\tau^\lambda\|$, λ, $\tau = 1, 2$, by the definition

$$\epsilon_1^1 = \epsilon_2^2 = 0 \qquad \epsilon_1^2 = \epsilon_2^1 = 1 \tag{3.5}$$

we see that we can write

$$\boldsymbol{\alpha}_s\psi_{\tau_1\ldots\tau_s\ldots\tau_n} = \sum_{\lambda=1}^{2}\epsilon_{\tau_s}^\lambda\boldsymbol{\sigma}_s\psi_{\tau_1\ldots\lambda\ldots\tau_n}. \tag{3.6}$$

We also have in (2.11) the product of operators $\boldsymbol{\sigma}_s B$ appearing in relation to the interaction term proportional to the position vector $\boldsymbol{r}$. Its effect on $\psi_{\tau_1\tau_2\ldots\tau_n}$ can be obtained by remembering that the matrix product of the operators has to be applied in reverse order, as indicated in equation (3.10) of [13]. Thus we arrive finally, from (2.11), at the fact that the components $\psi_{\tau_1\ldots\tau_n}$ of ψ satisfy the equation

$$\sum_{s=1}^{n}\sum_{\lambda=1}^{2}\{\epsilon_{\tau_s}^\lambda\boldsymbol{\sigma}_s\cdot[\boldsymbol{p} - \mathrm{i}\omega\boldsymbol{r}(-1)^{\lambda-\tau_s+\Delta}]\psi_{\tau_1\ldots\lambda\ldots\tau_n}\} = \left[nE - \sum_{s=1}^{n}(-1)^{1+\tau_s}\right]\psi_{\tau_1\ldots\tau_s\ldots\tau_n} \tag{3.7}$$

where

$$\Delta \equiv n + \tau_1 + \tau_2 + \cdots + \tau_n. \tag{3.8}$$

Note that if we want to pass from the Dirac oscillator to the free-particle problem, all we have to do is to put $\omega = 0$ in equation (3.7).

For $\omega > 0$ it is convenient to separate equations (3.7) into two sets depending on whether Δ of (3.8) is even or odd. For this purpose we start by defining creation and annihilation operators

$$\boldsymbol{\eta} = (1/\sqrt{2})(\omega^{1/2}\boldsymbol{r} - \mathrm{i}\omega^{-1/2}\boldsymbol{p}) \qquad \boldsymbol{\xi} = (1/\sqrt{2})(\omega^{1/2}\boldsymbol{r} + \mathrm{i}\omega^{-1/2}\boldsymbol{p}) \tag{3.9}$$

and also, to avoid the appearance of the imaginary factor i, we redefine the $\psi_{\tau_1\ldots\tau_n}$ terms of the wavefunction $\phi_{\tau_1\tau_2\ldots\tau_n}$ as

$$\psi_{\tau_1\tau_2\ldots\tau_n} \equiv \phi_{\tau_1\tau_2\ldots\tau_n} \qquad \text{for } \Delta \text{ even} \tag{3.10}$$

$$\psi_{\tau_1\tau_2\ldots\tau_n} \equiv -\mathrm{i}\phi_{\tau_1\tau_2\ldots\tau_n} \qquad \text{for } \Delta \text{ odd} \tag{3.11}$$

with Δ given by (3.8). We see then that equation (3.7) can be expressed in terms of the following two equations:

$$\sum_{s=1}^{n}\sum_{\lambda=1}^{2}\epsilon_{\tau_s}^{\lambda}[\sqrt{2\omega}(\boldsymbol{\sigma}_s\cdot\boldsymbol{\eta})]\phi_{\tau_1\ldots\lambda\ldots\tau_n} = \left[nE + \sum_{s=1}^{n}(-1)^{\tau_s}\right]\phi_{\tau_1\ldots\tau_s\ldots\tau_n} \tag{3.12}$$

$$\sum_{s=1}^{n}\sum_{\lambda=1}^{2}\epsilon_{\tau_s}^{\lambda}[\sqrt{2\omega}(\boldsymbol{\sigma}_s\cdot\boldsymbol{\xi})]\phi_{\tau_1\ldots\lambda\ldots\tau_n} = \left[nE + \sum_{s=1}^{n}(-1)^{\tau_s}\right]\phi_{\tau_1\ldots\tau_s\ldots\tau_n}. \tag{3.13}$$

In equation (3.12), $\phi_{\tau_1\ldots\tau_n}$ on the right-hand side corresponds to Δ even, while on the left-hand side is odd; in equation (3.13) the correlation is *vice versa*.

So far we have determined the equations (3.12) and (3.13) for the wavefunction $\phi_{\tau_1\ldots\tau_n}$, defined in equations (3.10) and (3.11), with the indexes τ_s, $s = 1, 2, \ldots, n$, which take the values $\tau_s = 1$ or 2. In the next subsection we will rewrite equations (3.12) and (3.13) in a more compact way, and introduce a recurrence procedure that allows their derivation in a simpler form for arbitrary n.

3.2. The matrix formulation of the wave equations

When dealing with $\phi_{\tau_1\tau_2\ldots\tau_n}$, with Δ of (3.8) either even or odd, it is useful to order them in a definite way. A convenient procedure for Δ even is to introduce first $\phi_{\tau_1\ldots\tau_n}$ in which all τ_s's equal 1, i.e. $\phi_{111\ldots1}$, and continue introducing the terms $\tau_s = 2$ in such a way that the indexes, taken together as a single number, increase all the time with Δ remaining even. For Δ odd we start with $\phi_{11\ldots12}$ and follow the same procedure. Thus, for example, when $n = 3$, we have the two sets

$$\boldsymbol{\Phi}_3^{+} \equiv \begin{bmatrix}\phi_{111}\\ \phi_{122}\\ \phi_{212}\\ \phi_{221}\end{bmatrix} \qquad \boldsymbol{\Phi}_3^{-} \equiv \begin{bmatrix}\phi_{112}\\ \phi_{121}\\ \phi_{211}\\ \phi_{222}\end{bmatrix} \tag{3.14}$$

where $\boldsymbol{\Phi}_3^{+}$, $\boldsymbol{\Phi}_3^{-}$ correspond to Δ even and odd, respectively, and the subindex 3 reminds us of the number $n = 3$, i.e. that our component wavefunctions are of the form $\phi_{\tau_1\tau_2\tau_3}$.

Our first objective will be to indicate how we can relate $\boldsymbol{\Phi}_n^{\pm}$ to $\boldsymbol{\Phi}_{n+1}^{\pm}$ so that later we can derive a recurrence procedure for the matrix representation of equations (3.12) and (3.13). We note that $\boldsymbol{\Phi}_n^{\pm}$ and $\boldsymbol{\Phi}_{n+1}^{\pm}$ have 2^{n-1} and 2^n components, respectively. We could get $\boldsymbol{\Phi}_{n+1}^{+}$ if we add an extra index 1 or 2 first *in* $\boldsymbol{\Phi}_n^{+}$ *or* $\boldsymbol{\Phi}_n^{-}$*, respectively*, and consider the set together in the order that they are mentioned. For $\boldsymbol{\Phi}_{n+1}^{-}$ we have to add an index 1 or 2 first *in* $\boldsymbol{\Phi}_n^{-}$ *or* $\boldsymbol{\Phi}_n^{+}$*, respectively*, and again consider the set together. This procedure, when applied, for example, to

$$\boldsymbol{\Phi}_2^{+} = \begin{bmatrix}\phi_{11}\\ \phi_{22}\end{bmatrix} \qquad \boldsymbol{\Phi}_2^{-} = \begin{bmatrix}\phi_{12}\\ \phi_{21}\end{bmatrix} \tag{3.15}$$

gives us immediately the set $\mathbf{\Phi}_3^{\pm}$ in (3.14). It is also clear that in this way the set of indexes, taken together as a single number, continues to increase as we go down the column defining $\mathbf{\Phi}_3^{\pm}$.

What is the matrix formulation of the set of the two equations (3.12) and (3.13) now? We start by defining certain submatrices that will be required, in a notation involving the indexes τ_s, i.e.

$$\mathsf{M}_n(v_1 \ldots v_n) \equiv \left\| \sum_{s=1}^{n} (\delta_{\tau_1}^{\bar{\tau}_1} \ldots \delta_{\tau_{s-1}}^{\bar{\tau}_{s-1}} v_s \epsilon_{\tau_s}^{\bar{\tau}_s} \delta_{\tau_{s+1}}^{\bar{\tau}_{s+1}} \ldots \delta_{\tau_n}^{\bar{\tau}_n}) \right\| \tag{3.16}$$

where v_s is defined by

$$v_s = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\eta}) \tag{3.17}$$

and if $\tau_1\tau_2 \ldots \tau_n$ corresponds to an even (odd) Δ of (3.8) then obviously, from the definition (3.5) of $\epsilon_{\tau_s}^{\bar{\tau}_s}$, we have that $\bar{\tau}_1\bar{\tau}_2 \ldots \bar{\tau}_n$ corresponds to an odd (even) Δ . For each of the two possibilities $\mathsf{M}_n(v_1 \ldots v_n)$ is a $2^{n-1} \times 2^{n-1}$ matrix. We further define the matrices

$$\mathsf{D}_n^{\pm} \equiv \left\| -\sum_{s=1}^{n} (-1)^{\tau_s} \delta_{\tau_1\tau_2\ldots\tau_n}^{\tau_1'\tau_2'\ldots\tau_n'} \right\| \tag{3.18}$$

where the $\pm$ sign corresponds to the situation when *both* $\tau_1\tau_2 \ldots \tau_n$ and $\tau_1'\tau_2' \ldots \tau_n'$ are associated with even or odd Δ. Again $\mathsf{D}_n^{\pm}$ are $2^{n-1} \times 2^{n-1}$ matrices but diagonal and numerical.

Now looking back to our equations (3.12) and (3.13) we see from the definitions of $\mathbf{\Phi}_n^{\pm}$ and (3.16) and (3.18) that they can be written in the matrix notation

$$\begin{bmatrix} \mathsf{D}_n^{+} & \mathsf{M}_n(v_1 \ldots v_n) \\ \mathsf{M}^{\dagger}(v_1 \ldots v_n) & \mathsf{D}_n^{-} \end{bmatrix} \begin{bmatrix} \mathbf{\Phi}_n^{+} \\ \mathbf{\Phi}_n^{-} \end{bmatrix} = nE \begin{bmatrix} \mathbf{\Phi}_n^{+} \\ \mathbf{\Phi}_n^{-} \end{bmatrix} \tag{3.19}$$

where $\mathsf{M}^{\dagger}$ is the Hermitian conjugate of M with $\boldsymbol{\xi}$ replacing $\boldsymbol{\eta}$.

From (3.18) the matrices $\mathsf{D}_n^{\pm}$ are very easily written down for the enumeration procedure we followed for the components of $\mathbf{\Phi}_n^{\pm}$. For the matrix $\mathsf{M}_n(v_1 \ldots v_n)$ the explicit determination is more complex and the best way of obtaining them seems to be a recurrence procedure.

For this purpose we propose to correlate matrices $\mathsf{M}_{n+1}(v_1 \ldots v_n v_{n+1})$ with $\mathsf{M}_n(v_1 \ldots v_n)$. From the enumeration procedure we outlined in the paragraph between equations (3.14) and (3.15) we see that in $\mathsf{M}_{n+1}(v_1 \ldots v_n v_{n+1})$, defined in a way similar to (3.16), the term

$$v_1 \epsilon_{\tau_1}^{\bar{\tau}_1} \delta_{\tau_2}^{\bar{\tau}_2} \ldots \delta_{\tau_{n+1}}^{\bar{\tau}_{n+1}} \tag{3.20}$$

changes $\tau_1 = 1$ or 2 into $\bar{\tau}_1 = 2$ or 1. Thus, in $\mathbf{\Phi}_{n+1}^{+}$ the terms coming from $\mathbf{\Phi}_n^{+}$ when we add to it an extra index 1 in the first position turns into terms coming from $\mathbf{\Phi}_n^{-}$ with an extra index 2 in the first position, and similarly when indexes 1 and 2 are interchanged. Thus, the term (3.20) transforms the first part of $\mathbf{\Phi}_{n+1}^{+}$ into the second and applies to it the operator v_1, and has a similar effect for $\mathbf{\Phi}_{n+1}^{-}$. The other terms in $\mathsf{M}_{n+1}(v_1 \ldots v_{n+1})$, i.e.

$$\delta_{\tau_1}^{\bar{\tau}_1} \ldots v_s \epsilon_{\tau_s}^{\bar{\tau}_s} \ldots \delta_{\tau_{n+1}}^{\bar{\tau}_{n+1}} \qquad s = 2, 3, \ldots n+1 \tag{3.21}$$

do not affect the first index in $\mathbf{\Phi}_{n+1}^{\pm}$ so they transform the first (second) part into the first (second) part but now acting with operator v_s that are increased by one position, i.e. $v_s \rightarrow v_{s+1}$ as compared with the situation for $\mathbf{\Phi}_n^{\pm}$. Thus we get the recurrence relation

$$\mathsf{M}_{n+1}(v_1 \ldots v_{n+1}) = \begin{bmatrix} \mathsf{M}_n(v_2 \ldots v_{n+1}) & v_1\mathsf{I} \\ v_1\mathsf{I} & \mathsf{M}_n(v_2 \ldots v_{n+1}) \end{bmatrix} \tag{3.22}$$

where $\mathbf{I}$ is the unit $2^{n-1}\times 2^{n-1}$ matrix and $\mathbf{M}_n$ has the same dimension while $\mathbf{M}_{n+1}(v_1 \ldots v_{n+1})$ is a $2^n \times 2^n$ matrix.

The same developments hold of course for the hermitian conjugate matrix $\mathbf{M}^{\dagger}_{n+1}(v_1 \ldots v_{n+1})$ only replacing $v_s = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\eta})$ by $v_s^{\dagger} = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\xi})$.

We are now in a position to write the matrix operator equation (3.19) explicitly starting with $n = 1$ and from it going to $n = 2$, and from the latter to $n = 3$ etc. The next point is to reduce equation (3.19) to a purely numerical one that allows us to get both the eigenvalues E and the eigenfunctions.

3.3. *Numerical matrix expression for our problem*

The first thing we notice in equation (2.11), and which holds also for its matrix form (3.19), is that it depends on a single position vector $\boldsymbol{r}$, and its corresponding momentum $\boldsymbol{p} = -\mathrm{i}\nabla$, but that it has many spin operators $\boldsymbol{\sigma}_s$, that appear in the $\boldsymbol{\alpha}_s$. As each of these spin operators is associated with the value $\frac{1}{2}$ of the corresponding observable, it suggests that equation (2.11) is in fact reducible in the spin and could have the following values of this observable,

$$\tfrac{n}{2}, \tfrac{n}{2} - 1, \ldots, \tfrac{1}{2} \text{ or } 0 \tag{3.23}$$

depending on whether n is even or odd. This is in fact the case, as mentioned after equation (2.6), when commenting on an observation of Kemmer [3].

It is possible to show, by a projection procedure on the permutation group, that we could restrict the most relevant component of $\phi_{\tau_1\tau_2\ldots\tau_n}$, i.e. $\phi_{11\ldots1}$ (corresponding to all the large components for positive energy) to the spin $(n/2)$. To implement this procedure we first note that, writing equation (3.19) explicitly, we can eliminate $\boldsymbol{\Phi}_n^-$ and get for $\boldsymbol{\Phi}_n^+$ the equation

$$\mathbf{M}_n(v_1 \ldots v_n)(nE\mathbf{I} - \mathbf{D}_n^-)^{-1}\mathbf{M}^{\dagger}(v_1 \ldots v_n)\boldsymbol{\Phi}_n^+ = (nE\mathbf{I} - \mathbf{D}_n^+)\boldsymbol{\Phi}_n^+ \tag{3.24}$$

which clearly commutes with the operator

$$\hat{N} = \boldsymbol{\eta} \cdot \boldsymbol{\xi}. \tag{3.25}$$

Also, as (3.24) contains only operators of the form $v_s = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\eta})$, $v_s^{\dagger} = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\xi})$, it also commutes with the total angular momentum operator

$$\boldsymbol{J} = \boldsymbol{L} + \boldsymbol{S} \tag{3.26}$$

where

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = -\mathrm{i}(\boldsymbol{\eta} \times \boldsymbol{\xi}) \tag{3.27}$$

and $\boldsymbol{S}$ is the total spin

$$\boldsymbol{S} = (\tfrac{1}{2})(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2 + \cdots + \boldsymbol{\sigma}_n) \tag{3.28}$$

where again the $\boldsymbol{\sigma}_s$ are given by direct products

$$\boldsymbol{\sigma}_s = I \otimes I \otimes \cdots \otimes I \otimes \boldsymbol{\sigma} \otimes I \otimes \cdots \otimes I \tag{3.29}$$

with $\boldsymbol{\sigma}$ in the sth position and I being a 2×2 unit matrix.

Thus the numbers N, $j(j+1)$, m corresponding to the eigenvalues of the operators $\hat{N}$, J^2, J_z characterize the solution of equation (3.19).

From the considerations of the previous paragraph it is clear that the components $\boldsymbol{\Phi}_n^+$ could be expressed in terms of the kets

$$|N(\ell, s)jm; wfr\rangle = \sum_{\mu,\sigma} \langle \ell\mu, s\sigma | jm\rangle R_{N\ell}(r) Y_{\ell\mu}(\theta, \varphi) \chi_{s\sigma}(wfr) \tag{3.30}$$

with $\langle\ |\ \rangle$ being a Clebsch–Gordan coefficient, $R_{n\ell}(r)$ the radial function of the oscillator, $Y_{\ell\mu}(\theta\varphi)$ a spherical harmonic and ℓ being the orbital angular momentum restricted by the conditions

$$|j-s| \leqslant \ell \leqslant j+s \qquad (-1)^N = (-1)^\ell. \tag{3.31}$$

The interesting part is the spin function $\chi_{s\sigma}(wfr)$ which is formed from n terms of value $\frac{1}{2}$. Clearly for $n > 2$ they require a further characterization and one way of doing this is to associate with them, apart from the value s of the spin, a partition $f \equiv [f_1 f_2 \ldots f_n]$, $f_1 + f_2 + \cdots + f_n = n$, corresponding to an irrep of the group of permutations of the spin indexes and also a Yamanouchi symbol r. However, sometimes a given partition f can appear more than once, and thus we introduce also the extra index w in the spin state.

For fixed N, j, m, the components of $\mathbf{\Phi}_n^+$ can then be written as

$$\sum_{\ell,s,w,f,r} a_{\tau_1\tau_2\ldots\tau_n}(\ell,s,w,f,r)|N(\ell,s)jm;wfr\rangle \tag{3.32}$$

where the indexes $\tau_1, \tau_2, \ldots, \tau_n$ correspond to an even Δ of (3.8). For $\mathbf{\Phi}_n^-$ we have, on the other hand, that it should have one quantum less than $\mathbf{\Phi}_n^+$ as operators of the type $v_s = \sqrt{2\omega}(\boldsymbol{\sigma}_s \cdot \boldsymbol{\eta})$ are applied to it and thus they can be written as

$$\sum_{\bar\ell,s,w,f,r} a_{\tau_1\tau_2\ldots\tau_n}(\bar\ell,s,w,f,r)|N-1(\bar\ell,s)jm;wfr\rangle \tag{3.33}$$

where the corresponding Δ of (3.8) is now odd.

We now note that equations (2.11) or (3.19) are invariant under permutations, but in the sense we discussed between equations (2.12) and (2.13). Thus any permutation P applied to our states (3.32) and (3.33) is really a product

$$P_\tau P_\sigma \tag{3.34}$$

where P_τ is the permutation of the indexes $\tau_1, \tau_2, \ldots, \tau_n$ of the coefficient a, while P_σ acts on the spin function $\chi_{s\sigma}(wfr)$ in the ket. The effect of the permutation on the coefficient indexes in a, is immediate, for example

$$[1,2]a_{\tau_1\tau_2\tau_3\ldots\tau_n} = a_{\tau_2\tau_1\tau_3\ldots\tau_n} \tag{3.35}$$

where $[1, 2]$ is the transposition indicated. On the other hand,

$$[1,2]\chi_{s\sigma}(wfr) = \sum_{\bar r} \chi_{s\sigma}(wf\bar r) D^f_{\bar r r}([1,2]) \tag{3.36}$$

where the matrix $\|D^f_{\bar r r}([1,2])\|$ is the representation of $[1, 2]$ characterized by the partition f.

We plan, in future publications, to project from our solutions the spin states associated with the *different* representations of the symmetric group. In this paper though we shall consider only the *symmetric* representation as it is the simplest to use and, for a given n, it gives the largest corresponding spin, i.e. $s = n/2$. To implement our objective we have to apply to our states the symmetrizer operator

$$\mathcal{P} \equiv \sum_P P_\tau P_\sigma \tag{3.37}$$

which is summed over all permutations.

To obtain the projected part in the simplest possible way it is convenient to note that the indexes τ_s are dichotomic variables. If we had substituted them by λ_s defined as

$\lambda_s = [\tau_s - \frac{3}{2}]$, then $\lambda_s = -\frac{1}{2}$ or $\frac{1}{2}$ when $\tau_s = 1$ or 2. Thus λ_s behave just as the spin variables that also take the values $\frac{1}{2}$ or $-\frac{1}{2}$. We could then make the transformation

$$a_{t\lambda w'f'r'} = \sum_{\tau_1\ldots\tau_n} \mathcal{M}_{t\lambda w'f'r',\tau_1\tau_2\ldots\tau_n} a_{\tau_1\tau_2\ldots\tau_n} \tag{3.38}$$

where $\mathcal{M}$ is a numerical orthogonal matrix of exactly the same form as in the spin case. In (3.38) t, λ play the role that s, σ have for the spin case, and $w'f'r'$ have for the τ indexes the same meaning that w, f, r had for the spin indexes. In this way the components of $\Phi_n^{\pm}$ can be written as

$$\phi_{t\lambda w'f'r'} = \sum_{\ell,s,w,f,r} a_{t\lambda w'f'r'}(\ell, s, w, f, r)|N(\ell, s)jm; wfr\rangle \tag{3.39}$$

$$\phi_{t\lambda w'f'r'} = \sum_{\ell,s,w,f,r} a_{t\lambda w'f'r'}(\bar{\ell}, s, w, f, r)|N-1(\bar{\ell}, s)jm; wfr\rangle \tag{3.40}$$

where λ in (3.39) and (3.40) are respectively selected in such a way that they come from $\tau_1\tau_2\ldots\tau_n$ corresponding to even or odd Δ of (3.8). In fact from the relation $\lambda_s = [\tau_s - \frac{3}{2}]$ mentioned above we have $\lambda = [\Delta - (5n/2)]$.

If the full matrix appearing on the left-hand side of (3.19) is denoted by $\mathcal{D}$, and the full wavefunction associated with (3.39) and (3.40) by $\bar{\Psi}$, while the matrix whose elements appear in (3.38) is written as $\mathcal{M}$, then equation (3.19) becomes

$$\widetilde{\mathcal{M}}\mathcal{D}\mathcal{M}\bar{\Psi} = nE\bar{\Psi} \tag{3.41}$$

with $\sim$ indicating a transposition.

So far we have given our functions ϕ of (3.39) and (3.40) in their most general form but, as we have said before, we are only interested in the ones which are projected to the symmetric representation $[n]$ of the permutation group, i.e. in the wavefunction resulting from the application to the ϕ's of (3.39) and (3.40) of the projection operator of (3.37). From standard representation theory [14] we then find, for example, for the term in (3.39) that

$$\begin{aligned}
\mathcal{P}\phi_{t\lambda w'f'r'} &= \sum_P \sum_{\ell,s,w,f,r} \{[P_\tau a_{t\lambda w'f'r'}(\ell, s, w, f, r)][P_\sigma|N(\ell, s)jm; wfr\rangle]\} \\
&= \sum_{\bar{r}'\bar{r}} \sum_{\ell,s,w,f,r} \left\{ a_{t\lambda w'f'\bar{r}'}(\ell, s, w, f, r)|N(\ell, s)jm; wf\bar{r}\rangle \sum_P D^{f'}_{\bar{r}'r'}(P) D^{f}_{\bar{r}r}(P) \right\} \\
&= \frac{n!}{d_f} \delta_{ff'}\delta_{rr'} \sum_{\ell,s,w,f,\bar{r}} a_{t\lambda w'f\bar{r}}(\ell, s, w, f, r)|N(\ell, s)jm; wf\bar{r}\rangle
\end{aligned} \tag{3.42}$$

where we used the well known relation [14]

$$\sum_P D^{f'}_{\bar{r}'r'}(P) D^{f*}_{\bar{r}r}(P) = \frac{n!}{d_f}\delta_{f'f}\delta_{r'r}\delta_{\bar{r}'\bar{r}} \tag{3.43}$$

where $n!$ is the number of elements in our permutation group and d_f the dimension of the representation characterized by the partition f. Note that as the unitary representations of the permutation group are real, the conjugate symbol $*$ in (3.43) can be suppressed and this was the form used to derive the right-hand side of (3.42). A similar relation to (3.42) holds when we apply $\mathcal{P}$ to the ϕ of (3.40) instead of (3.39).

Note that if in the projected representations we deal with $f' = [n]$, i.e. the symmetric representation of the permutation group for the coefficient a, we see from (3.42) that the spin part also corresponds to $f = [n]$, and this can only be achieved if the total spin is $n/2$ as, for its highest projection, we have

$$\chi_{\frac{n}{2}\frac{n}{2}w[n]} = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \cdots \otimes \begin{pmatrix}1\\0\end{pmatrix} \tag{3.44}$$

which is obviously symmetric under permutations.

Thus, in the old notation, the $\phi_{111\ldots1}$ which corresponds to the part involving only large components when the energy is positive, as it is symmetric under exchange of the τ indexes, can *only* have spin $(n/2)$, and this is why we have given the present paper the title of Dirac oscillator with arbitrary spin, as our formalism can be implemented, in principle, for any n.

To reduce the matrix operator equation to its numerical form, we will then only need to deal with matrix elements all of which can be reduced to the form

$$\begin{aligned}\langle N(\ell, s) jm, wfr|\sqrt{2\omega}\boldsymbol{\sigma}_t \cdot \boldsymbol{\eta}|N-1(\bar{\ell}, \bar{s}) jm, \bar{w}\bar{f}\bar{r}\rangle \\ &= (-1)^{\ell+\bar{s}-j} W(\bar{\ell}\ell\bar{s}s; 1j)[(2\ell+1)(2s+1)]^{1/2}\sqrt{2\omega}\langle N\ell\|\eta\|N-1\bar{\ell}\rangle \\ &\quad \times\langle s, wfr\|\sigma_t\|\bar{s}, \bar{w}\bar{f}\bar{r}\rangle\end{aligned} \tag{3.45}$$

where we have used a well known relation of Rose [15], with W being a Racah coefficient, and where the reduced matrix element of the η has the expression [16]

$$\langle N\ell\|\eta\|N-1\bar{\ell}\rangle = \left[\frac{(N+\ell+1)\ell}{2\ell+1}\right]^{1/2}\delta_{\bar{\ell}\ell-1} + \left[\frac{(N-\ell)(\ell+1)}{2\ell-1}\right]^{1/2}\delta_{\bar{\ell}\ell+1}. \tag{3.46}$$

The reduced matrix element of σ_t is trivial for $n = 1$ or 2, but for larger values it requires a more complex Racah algebra analysis, which will be indicated in the specific cases to be considered later.

As a last point in our general discussion we notice that in (3.24) parity would also be a good quantum number as the equation remains invariant if we interchange $\boldsymbol{\eta}$, $\boldsymbol{\xi}$, with $-\boldsymbol{\eta}$, $-\boldsymbol{\xi}$. Thus in our procedure we will have two cases: one in which we start with $\ell = j + (n/2)$, so N and $j + (n/2)$ have the same parity, and we include all the other ℓ's by subtracting multiples of 2; the other when we start with $\ell = j + (n/2) - 1$ and a similar analysis holds.

All the general steps we have outlined in this section will become clearer as we discuss the examples with $n = 1, 2, 3$ in the following section.

4. Examples

Our procedure obviously does not apply to the case of spin 0 as it would imply $n = 0$, but this case has already been discussed in [10] using Kemmer's [3] formalism and, in our units, gives the expression

$$E^2 = 1 + 2\omega N \tag{4.1}$$

with the wavefunction being that of the standard harmonic oscillator, i.e. equation (3.30) with $n = 0$. Note that in the following analysis we will be interested only in *positive energies*, which will be close to 1 if $\omega \ll 1$ and significant as they have no contribution from the negative energy part. Thus for spin 0 the energy we shall consider is

$$E = (1 + 2\omega N)^{1/2} \simeq 1 + \omega N \tag{4.2}$$

where the right-hand term holds in the non-relativistic limit, i.e. when, in our units, $\omega \ll 1$.

Our starting point will then be $n = 1$.

4.1. Spin $\frac{1}{2}$

Taking into account the definition of the matrices in (3.16) and (3.18) it is clear that for $n = 1$ they become 1×1 matrices and equation (3.19) takes the form

$$\begin{bmatrix} 1 & v_1 \\ v_1^\dagger & -1 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} = E \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} \tag{4.3}$$

where $\boldsymbol{\Phi}_1^+$, $\boldsymbol{\Phi}_1^-$ also have one component, and, following the enumeration procedure discussed between equations (3.14) and (3.15), they could be denoted by ϕ_1, ϕ_2 as they appear in (4.3). From equation (3.17) $v_1 = \sqrt{2\omega}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\eta})$, $v_1^\dagger = \sqrt{2\omega}(\boldsymbol{\sigma}_1 \cdot \boldsymbol{\xi})$.

To find the values of the energy we have to discuss the two alternatives mentioned at the end of section 3. In the first case

$$\phi_1 = a_1|+\rangle \qquad \phi_2 = a_2|-) \tag{4.4}$$

while in the second

$$\phi_1 = a_1'|-\rangle \qquad \phi_2 = a_2'|+) \tag{4.5}$$

where we shall use the short-hand notation

$$|\pm\rangle = |N(j \pm \tfrac{1}{2}, \tfrac{1}{2})jm\rangle \qquad |\pm) = |N-1(j \pm \tfrac{1}{2}, \tfrac{1}{2})jm\rangle \tag{4.6}$$

noting that we employ angular kets when the number of quanta is N, and round kets when it is $N - 1$.

Substituting expressions (4.4) into (4.3), and multiplying the equations resulting from the first and second rows by the bra's $\langle+|$, $(-|$, respectively, we get the numerical matrix equation

$$\begin{bmatrix} 1 & \langle+|v_1|-) \\ (-|v_1^\dagger|+\rangle & -1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}. \tag{4.7}$$

From hermiticity considerations $(-|v_1^\dagger|+\rangle = \langle+|v_1|-)$, and since the latter was calculated in (3.45) and (3.46) we only have to add the fact that

$$\langle\tfrac{1}{2}\|\sigma_1\|\tfrac{1}{2}\rangle = \frac{\langle\frac{1}{2}\frac{1}{2}|\sigma_{01}|\frac{1}{2}\frac{1}{2}\rangle}{\langle\frac{1}{2}\frac{1}{2}, 10|\frac{1}{2}\frac{1}{2}\rangle} = \sqrt{\tfrac{3}{4}} \tag{4.8}$$

where the denominator is a Clebsch–Gordan coefficient and the numerator is obviously 1. Thus we obtain

$$\langle+|v_1|-) = -\sqrt{2\omega}(N + j + \tfrac{3}{2})^{1/2}. \tag{4.9}$$

A similar analysis, when we substitute (4.5) into (4.3), gives the numerical matrix equation

$$\begin{bmatrix} 1 & \langle-|v_1|+) \\ (+|v_1^\dagger|-\rangle & -1 \end{bmatrix} \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} = E \begin{bmatrix} a_1' \\ a_2' \end{bmatrix} \tag{4.10}$$

where

$$(+|v_1^\dagger|-\rangle = \langle-|v_1|+) = \sqrt{2\omega}(N - j + \tfrac{1}{2})^{1/2}. \tag{4.11}$$

From (4.7) and (4.10) we get, respectively, for the energy the secular equations

$$E^2 = 1 + 2\omega(N + j + \tfrac{3}{2}) \tag{4.12}$$

$$E^2 = 1 + 2\omega(N - j + \tfrac{1}{2}) \tag{4.13}$$

which are precisely the results derived in [5] for the Dirac oscillator of spin $\frac{1}{2}$, but now obtained through a procedure that can be applied to any values of the spin.

As discussed in [5] and [17], the values of $(2\omega)^{-1}(E^2 - 1)$ have a finite or infinite degeneracy, depending on whether we consider the expression (4.12) or (4.13), respectively, and they are plotted in figure 1. These degeneracies can be explained by an SO(4) or SO(3, 1) symmetry group, respectively, as discussed in [17].

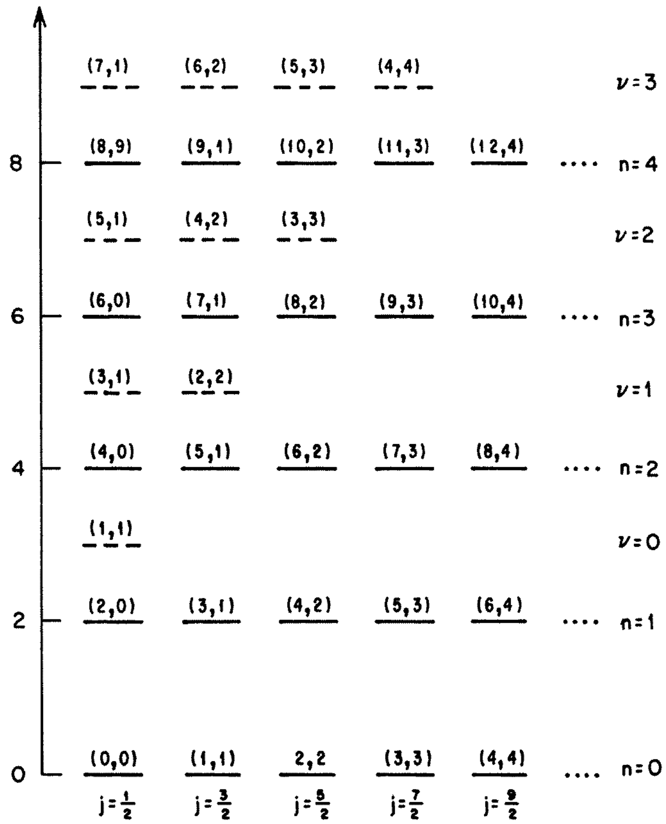

Figure 1. Values of E^2 as a function of the number of quanta N and the total angular momentum j for spin $s = \frac{1}{2}$. In the abscissa we give the values of j, whereas in the ordinate those of ν, n defined as follows: $\nu = (E^2 - 1)/4\omega - \frac{3}{2}$, when E^2 is given by (4.12); $n = (E^2 - 1)/4\omega$, when E^2 is given by (4.13). On top of each energy level we give the values of N, ℓ. The values of ν are indicted by a dashed line and those of n by a full line. The former are finite while the later are infinitely degenerate. Taken from [17] by Quesne and Moshinsky.

4.2. Spin 1

Using the recurrence relation of equation (3.22) and the definitions (3.16) and (3.18) of the matrices $\mathbf{M}_n$ and $\mathbf{D}_n^{\pm}$, as well as the enumeration procedure discussed between equations (3.14) and (3.15), we can immediately write for spin 1 equation (3.19) in the form

$$\begin{bmatrix} 2 & 0 & v_2 & v_1 \\ 0 & -2 & v_1 & v_2 \\ v_2^{\dagger} & v_1^{\dagger} & 0 & 0 \\ v_1^{\dagger} & v_2^{\dagger} & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11} \\ \phi_{22} \\ \phi_{12} \\ \phi_{21} \end{bmatrix} = 2E \begin{bmatrix} \phi_{11} \\ \phi_{22} \\ \phi_{12} \\ \phi_{21} \end{bmatrix}. \tag{4.14}$$

From (3.38) it is convenient to write $\phi_{\tau_1\tau_2}$ in terms of irreducible representations of the permutation group of two indexes as

$$\phi_{[2]} \equiv \frac{1}{\sqrt{2}}(\phi_{12} + \phi_{21}) \qquad \phi_{[11]} \equiv \frac{1}{\sqrt{2}}(\phi_{12} - \phi_{21}). \tag{4.15}$$

For ϕ_{11}, ϕ_{22}, while they are not required by the permutation group as they correspond to its symmetric representation, it is also convenient to carry out the same transformation, i.e.

$$\phi_{+} \equiv \frac{1}{\sqrt{2}}(\phi_{11} + \phi_{22}) \qquad \phi_{-} \equiv \frac{1}{\sqrt{2}}(\phi_{11} - \phi_{22}). \tag{4.16}$$

Thus we can write

$$\begin{bmatrix} \phi_{11} \\ \phi_{22} \\ \phi_{12} \\ \phi_{21} \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \phi_{+} \\ \phi_{-} \\ \phi_{[2]} \\ \phi_{[11]} \end{bmatrix} \tag{4.17}$$

and from (4.14) the vector on the right-hand side satisfies the equation

$$\begin{bmatrix} 0 & 2 & v_1 + v_2 & 0 \\ 2 & 0 & 0 & v_2 - v_1 \\ v_1^\dagger + v_2^\dagger & 0 & 0 & 0 \\ 0 & v_2^\dagger - v_1^\dagger & 0 & 0 \end{bmatrix} \begin{bmatrix} \phi_{+} \\ \phi_{-} \\ \phi_{[2]} \\ \phi_{[11]} \end{bmatrix} = 2E \begin{bmatrix} \phi_{+} \\ \phi_{-} \\ \phi_{[2]} \\ \phi_{[11]} \end{bmatrix}. \tag{4.18}$$

Now using the following short-hand notation for the kets

$$|\pm\rangle \equiv |N(j \pm 1, 1)jm\rangle \qquad |1\rangle \equiv |N(j, 1)jm\rangle \qquad |0\rangle = |N(j, 0)jm\rangle \tag{4.19}$$

and similar ones with a *round* ket when we have $N - 1$ instead of N, we can write the ϕ functions in (4.18) as

$$\begin{aligned} &\phi_{+} = a_{++}|+\rangle + a_{+-}|-\rangle \qquad \phi_{-} = a_{-+}|+\rangle + a_{--}|-\rangle \\ &\phi_{[2]} = a_{[2]}|1) \qquad \phi_{[11]} = a_{[11]}|0). \end{aligned} \tag{4.20}$$

Substituting (4.20) into (4.18) and multiplying by the bra's $\langle\pm|$, $(1|$, $(0|$ as required in the corresponding row, we see that the matrix operator equation (4.18) becomes the numerical operator

$$\begin{bmatrix} -2E & 0 & 2 & 0 & M_{+1} & 0 \\ 0 & -2E & 0 & 2 & M_{-1} & 0 \\ 2 & 0 & -2E & 0 & 0 & M_{+0} \\ 0 & 2 & 0 & -2E & 0 & M_{-0} \\ M_{+1} & M_{-1} & 0 & 0 & -2E & 0 \\ 0 & 0 & M_{+0} & M_{-0} & 0 & -2E \end{bmatrix} \begin{bmatrix} a_{++} \\ a_{+-} \\ a_{-+} \\ a_{--} \\ a_{[2]} \\ a_{[11]} \end{bmatrix} = 0 \tag{4.21}$$

where, noting that

$$v_1 + v_2 = 2\sqrt{2\omega}(\boldsymbol{S} \cdot \boldsymbol{\eta}) \qquad v_2 - v_1 = 2\sqrt{2\omega}(\boldsymbol{S}' \cdot \boldsymbol{\eta}) \tag{4.22}$$

with

$$\boldsymbol{S} = \tfrac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2) \qquad \boldsymbol{S}' = \tfrac{1}{2}(\boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1) \tag{4.23}$$

we see that the symbols appearing in (4.21) have the values

$$\begin{aligned} M_{+1} &= \langle +|2\sqrt{2\omega}(\boldsymbol{S} \cdot \boldsymbol{\eta})|1) \\ &= -2\sqrt{2\omega}[j(N + j + 2)/(2j + 1)]^{1/2} \end{aligned} \tag{4.24}$$

$$\begin{aligned} M_{-1} &= \langle -|2\sqrt{2\omega}(\boldsymbol{S} \cdot \boldsymbol{\eta})|1) \\ &= 2\sqrt{2\omega}[(j + 1)(N - j + 1)/(2j + 1)]^{1/2} \end{aligned} \tag{4.25}$$

$$M_{+0} = \langle +|2\sqrt{2\omega}(\boldsymbol{S}' \cdot \boldsymbol{\eta})|0)$$

$$= 2\sqrt{2\omega}[(j+1)(N+j+2)/(2j+1)]^{1/2} \tag{4.26}$$

$$M_{-0} = \langle -|2\sqrt{2\omega}(\boldsymbol{S}'\cdot\boldsymbol{\eta})|0\rangle$$
$$= 2\sqrt{2\omega}[j(N-j+1)/(2j+1)]^{1/2}. \tag{4.27}$$

Substituting (4.24)–(4.27) into (4.21) we then have a 6×6 matrix containing $-2E$ in the diagonal and numerical expressions, and functions of ω, N, j, in the other terms, which leads immediately, from equating to zero the determinant of the matrix in (4.21), to the secular equation

$$\{E^2[E^2-1-2\omega(N+1)][E^2-1-2\omega(N+2)]-4\omega^2 j(j+1)\}=0. \tag{4.28}$$

We still need to discuss the second type of state mentioned at the end of section 3 where we start with $\ell = j + n/2 - 1$ which is equal to j as $n = 2$.

In this case ϕ_+, ϕ_- can only be expressed in terms of what we called in (4.19) the ket $|1\rangle$. As $\phi_{[2]}$, $\phi_{[11]}$ have opposite parity they have to be expressed in terms of the ket $|\pm)$, and as the latter correspond to spin 1, they are symmetric under spin exchange, and only the term $\phi_{[2]}$ survives. Thus equation (4.18) becomes

$$\begin{bmatrix} -2E & 2 & v_1+v_2 & 0 \\ 2 & -2E & 0 & v_2-v_1 \\ v_1^\dagger+v_2^\dagger & 0 & -2E & 0 \\ 0 & v_2^\dagger-v_1^\dagger & 0 & -2E \end{bmatrix} \begin{bmatrix} a_+|1\rangle \\ a_-|1\rangle \\ a_{[2]}|+)+b_{[2]}|-) \\ 0 \end{bmatrix} = 0. \tag{4.29}$$

Multiplying appropriately on the left-hand side by the bras $\langle 1|$, $(+|$, $(-|$ we get the numerical equations

$$\begin{aligned} &-2Ea_+ + 2a_- + \langle 1|v_1+v_2|+)a_{[2]} + \langle 1|v_1+v_2|-)b_{[2]} = 0 \\ &2a_+ - 2Ea_- = 0 \\ &(+|v_1^\dagger + v_2^\dagger|1\rangle a_+ - 2Ea_{[2]} = 0 \\ &(-|v_1^\dagger + v_2^\dagger|1\rangle a_- - 2Eb_{[2]} = 0. \end{aligned} \tag{4.30}$$

From the last three equations we can substitute $a_{[2]}$, $b_{[2]}$, a_- in the first and using the hermiticity condition we obtain

$$\{-4E^2 + 4 + [\langle 1|v_1+v_2|+)]^2 + [\langle 1|v_1+v_2|-)]^2\}a_+ = 0 \tag{4.31}$$

so that from (4.24) and (4.25) we get the square of the energy as

$$E^2 = 1 + 2\omega(N+1). \tag{4.32}$$

We now examine equations (4.28) and (4.32). If $\omega \ll 1$ we can disregard the $\omega^2 j(j+1)$ term in (4.28) so thus the positive energies become

$$E \simeq 1+\omega(N+1) \qquad E \simeq 1+\omega(N+2) \qquad E \simeq 0 \tag{4.33}$$

while from (4.32) we also get the first relation in (4.33). The value $E = 0$ is an infinitely degenerate state, as all the kets independently of the values of ω, N, j are eigenstates. They correspond to what we have called a cockroach nest [18] and have no physical significance as they involve both positive and negative energy states. The states close to 1 for $\omega \ll 1$ are, however, of physical interest, and thus it is worthwhile following their behaviour as a function of ω and N. This is immediate for (4.32), but for (4.28) we have to solve a third degree algebraic equation in E^2 and thus this gives rise to a complicated spectrum.

4.3. Spin $\frac{3}{2}$

Using the recurrence relation of equation (3.22) and the definitions (3.16) and (3.18) of the matrices $\mathbf{M}_n$ and $\mathbf{D}_n^{\pm}$, when $n = 3$, as well as the enumeration procedure discussed between equations (3.14) and (3.15), we can immediately write for spin $\frac{3}{2}$ equation (3.19) in the form

$$\begin{bmatrix} 3-3E & 0 & 0 & 0 & v_3 & v_2 & v_1 & 0 \\ 0 & -1-3E & 0 & 0 & v_2 & v_3 & 0 & v_1 \\ 0 & 0 & -1-3E & 0 & v_1 & 0 & v_3 & v_2 \\ 0 & 0 & 0 & -1-3E & 0 & v_1 & v_2 & v_3 \\ v_3^\dagger & v_2^\dagger & v_1^\dagger & 0 & 1-3E & 0 & 0 & 0 \\ v_2^\dagger & v_3^\dagger & 0 & v_1^\dagger & 0 & 1-3E & 0 & 0 \\ v_1^\dagger & 0 & v_3^\dagger & v_2^\dagger & 0 & 0 & 1-3E & 0 \\ 0 & v_1^\dagger & v_2^\dagger & v_3^\dagger & 0 & 0 & 0 & -3-3E \end{bmatrix} \begin{bmatrix} \phi_{111} \\ \phi_{122} \\ \phi_{212} \\ \phi_{221} \\ \phi_{112} \\ \phi_{121} \\ \phi_{211} \\ \phi_{222} \end{bmatrix} = 0. \tag{4.34}$$

To convert this matrix operator equation into numerical form we have to use the developments (3.32), (3.33) which again appear, as indicated at the end of section 3, in two forms, although for brevity in the case of spin $\frac{3}{2}$ we shall restrict ourselves to the first form, i.e. we start with $\ell = j + \frac{3}{2}$. We use the compact notation

$$|+\rangle \equiv |N(j+\tfrac{3}{2},\tfrac{3}{2})jm\rangle \qquad |-\rangle \equiv |N(j-\tfrac{1}{2},\tfrac{3}{2})jm\rangle \tag{4.35}$$

$$|1\rangle \equiv |N(j-\tfrac{1}{2},\tfrac{1}{2})jm,1\rangle \qquad |0\rangle \equiv |N(j-\tfrac{1}{2},\tfrac{1}{2})jm,0\rangle \tag{4.36}$$

where in the full notation of (3.30) we have suppressed the index w which is irrelevant and the partition [3], [21] associated with the spins $\frac{3}{2}$ and $\frac{1}{2}$, respectively. However, in the latter case we have left the Yamanouchi symbol, which we designate by 1 and 0, that corresponds to the full value of the first two spins. All the above kets will be denoted as angular as they correspond to N quanta. For the kets with $N-1$ quanta we shall use a round ket notation and, because they have opposite parity, they will be denoted as

$$|+) \equiv |N-1(j+\tfrac{1}{2},\tfrac{3}{2})jm\rangle \qquad |-) \equiv |N-1(j-\tfrac{3}{2},\tfrac{3}{2})jm\rangle \tag{4.37}$$

$$|1) \equiv |N-1(j+\tfrac{1}{2},\tfrac{1}{2})jm,1\rangle \qquad |0) \equiv |N-1(j+\tfrac{1}{2},\tfrac{1}{2})jm,0\rangle. \tag{4.38}$$

Now we shall make use of the notations (3.38), (3.39) and (3.40) to transform the states $\phi_{\tau_1\tau_2\tau_3}$ into $\phi^{\pm}_{f'r'}$ where w is unnecessary and the partition f' already indicates the value of the index t as we remarked above. λ is replaced by the $\pm$ above the ϕ. Thus, from well known considerations [19], we have

$$\begin{pmatrix} \phi_{111} \\ \phi_{122} \\ \phi_{212} \\ \phi_{221} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \end{pmatrix} \begin{pmatrix} \phi_{111} \\ \phi^{+}_{[3]} \\ \phi^{+}_{[21]1} \\ \phi^{+}_{[21]0} \end{pmatrix} \tag{4.39}$$

while

$$\begin{pmatrix} \phi_{112} \\ \phi_{121} \\ \phi_{211} \\ \phi_{222} \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \phi^{-}_{[21]0} \\ \phi^{-}_{[21]1} \\ \phi^{-}_{[3]} \\ \phi_{222} \end{pmatrix}. \tag{4.40}$$

Substituting the eight component vector in (4.34) by the relation indicated in (4.39) and (4.40) we obtain the equation

$$\begin{bmatrix} 3-3E & 0 & 0 & 0 & \gamma & -\beta & \alpha & 0 \\ 0 & -1-3E & 0 & 0 & -\frac{1}{\sqrt{3}}\gamma & \frac{1}{\sqrt{3}}\beta & \frac{2}{\sqrt{3}}\alpha & \alpha \\ 0 & 0 & -1-3E & 0 & -\sqrt{\frac{2}{3}}\gamma & \delta & -\frac{1}{\sqrt{3}}\beta & \beta \\ 0 & 0 & 0 & -1-3E & \epsilon & -\sqrt{\frac{2}{3}}\gamma & \frac{1}{\sqrt{3}}\gamma & -\gamma \\ \gamma^\dagger & -\frac{1}{\sqrt{3}}\gamma^\dagger & -\sqrt{\frac{2}{3}}\gamma^\dagger & \epsilon^\dagger & 1-3E & 0 & 0 & 0 \\ -\beta^\dagger & \frac{1}{\sqrt{3}}\beta^\dagger & \delta^\dagger & -\sqrt{\frac{2}{3}}\gamma^\dagger & 0 & 1-3E & 0 & 0 \\ \alpha^\dagger & \frac{2}{\sqrt{3}}\alpha^\dagger & -\frac{1}{\sqrt{3}}\beta^\dagger & \frac{1}{\sqrt{3}}\gamma^\dagger & 0 & 0 & 1-3E & 0 \\ 0 & \alpha^\dagger & \beta^\dagger & -\gamma^\dagger & 0 & 0 & 0 & -3-3E \end{bmatrix} \begin{bmatrix} \phi_{111} \\ \phi^+_{[3]} \\ \phi^+_{[21]1} \\ \phi^+_{[21]0} \\ \phi^-_{[21]0} \\ \phi^-_{[21]1} \\ \phi^-_{[3]} \\ \phi_{222} \end{bmatrix} = 0 \qquad (4.41)$$

where the Greek letters are given by

$$\begin{aligned} &\alpha = \tfrac{1}{\sqrt{3}}(v_1 + v_2 + v_3) \qquad \beta = \tfrac{1}{\sqrt{6}}(2v_3 - v_1 - v_2) \\ &\gamma = \tfrac{1}{\sqrt{2}}(v_1 - v_2) \qquad \delta = \tfrac{2}{3}(v_1 + v_2) - \tfrac{1}{3}v_3 \qquad \epsilon = v_3 \end{aligned} \qquad (4.42)$$

with the same definition for $\alpha^\dagger$, $\beta^\dagger$, $\gamma^\dagger$, $\delta^\dagger$, $\epsilon^\dagger$ with v_s replaced by $v_s^\dagger$.

We note that the matrix in equation (4.41) differs from (4.34) only by a numerical orthogonal transformation and thus the former must also be invariant under permutations $P_\tau P_\sigma$ as the latter was derived from equation (2.11) which has this property. Thus if we write equation (4.41) in the compact form

$$(\mathcal{M} - 3E\mathcal{I})\mathbf{\Phi} = 0 \qquad (4.43)$$

we have from the fact that

$$P_\tau P_\sigma \mathcal{M} (P_\tau P_\sigma)^{-1} = \mathcal{M} \qquad (4.44)$$

that $P_\tau P_\sigma \mathbf{\Phi}$ is also a solution of equation (4.43).

We can then sum all of these solutions to conclude that

$$(\mathcal{M} - 3E\mathcal{I})\mathcal{P}\mathbf{\Phi} = 0 \qquad (4.45)$$

where $\mathcal{P}$ is the symmetrizer of (3.37). Thus the components of our state are invariant under permutation.

Taking into account the definitions (4.35)–(4.40) for the kets and their coefficients, as well as the effect of $\mathcal{P}$ on the ϕ indicated in (3.42), we conclude that the components of $\mathcal{P}\phi$ can only be of the form indicated below

$$\begin{aligned} &\mathcal{P}\phi_{111} = a_{111}|+\rangle + b_{111}|-\rangle \qquad \mathcal{P}\phi^+_{[3]} = a^+_{[3]}|+\rangle + b^+_{[3]}|-\rangle \\ &\mathcal{P}\phi^+_{[21]1} = a^+_{[21]1}|1\rangle + a^+_{[21]0}|0\rangle \qquad \mathcal{P}\phi^+_{[21]0} = b^+_{[21]0}|0\rangle + b^+_{[21]1}|1\rangle \\ &\mathcal{P}\phi^-_{[21]0} = b^-_{[21]0}|0) + b^-_{[21]1}|1) \qquad \mathcal{P}\phi^-_{[21]1} = a^-_{[21]1}|1) + a^-_{[21]0}|0) \\ &\mathcal{P}\phi^-_{[3]} = a^-_{[3]}|+) + b^-_{[3]}|-) \qquad \mathcal{P}\phi_{222} = a_{222}|+) + b_{222}|-). \end{aligned} \qquad (4.46)$$

We have used, as before, both the letters a and b if they are required to distinguish different terms in (4.46).

If we substitute (4.46) into (4.41) and multiply appropriately the left-hand sides by the bras $\langle +|$, $\langle -|$, $\langle 1|$, $\langle 0|$; $(+|$, $(-|$, $(1|$, $(0|$, we obtain a system of 16 linear equations in the coefficients (a_{111}, b_{111}, $a^+_{[3]}$, $b^+_{[3]}$, $a^+_{[21]1}$, $a^+_{[21]0}$, $b^+_{[21]1}$, $b^+_{[21]0}$, $b^-_{[21]1}$, $b^-_{[21]0}$, $a^-_{[21]1}$, $a^-_{[21]0}$,

$a^-_{[3]}$, $b^-_{[3]}$, a_{222}, b_{222}), and the corresponding matrix can be written in the form given in the appendix, where the symbols are the functions of ω, N, j given below,

$$c \equiv -\left[\frac{(2\omega)(N+j+\frac{5}{2})(2j+3)}{2(j+1)}\right]^{1/2} \qquad d \equiv -\left[\frac{(2\omega)j(N+j+\frac{5}{2})}{(j+1)}\right]^{1/2} \tag{4.47}$$

$$e \equiv -\left[\frac{(2\omega)j(2j-1)(N-j+\frac{1}{2})}{6(j+1)(j-1)}\right]^{1/2} \qquad g \equiv \left[\frac{(2\omega)(2j-1)(N-j+\frac{1}{2})(2j+3)}{3(j+1)(j-1)}\right]^{1/2} \tag{4.48}$$

$$h \equiv -\left[\frac{(2\omega)(N+j+\frac{1}{2})(j+1)}{j}\right]^{1/2} \qquad q \equiv \left[\frac{(2\omega)j(N-j+\frac{1}{2})}{(j-1)}\right]^{1/2} \tag{4.49}$$

$$u \equiv \left[\frac{(2\omega)(2j+3)(N-j+\frac{1}{2})}{j}\right]^{1/2} \qquad y \equiv -\left[\frac{(2\omega)(2j-1)(N+j+\frac{1}{2})}{j}\right]^{1/2}. \tag{4.50}$$

They are determined from (3.45) and (3.46) together with the fact that

$$\langle s\sigma f r|\sigma_{0t}|\bar{s}\sigma\bar{f}\bar{r}\rangle \tag{4.51}$$

can be obtained straightforwardly by expressing $|s\sigma f r\rangle$ in terms of products of spin $\frac{1}{2}$ functions [19].

We found that mathematics could not give the eigenvalues or even the determinant of the matrix in the appendix in a symbolic form. Thus the values of E have to be calculated numerically for fixed ω, N, j. In section 5 we shall only discuss qualitatively the behaviour of E^2.

5. Conclusion

The reader may find it strange that we started this paper with a discussion of five articles that are over 50 years old, and then proceeded to make an analysis of the problem with techniques developed mainly by the authors and their collaborators in the last few years.

The reason for this is that while the problem of a relativistic particle with arbitrary spin has given rise to an extensive literature, our approach is quite different from that followed by other authors and thus we mentioned only the literature we actually used.

In our analysis we start with a single relativistic free-particle equation with arbitrary spin, in analogy with what was done for a system of many spin $\frac{1}{2}$ particles by Moshinsky *et al* [12], following an approach by Barut [11].

We need, however, to impose the constraint that the wavefunction is invariant under permutation of the spin indices if we wish the large component of our wavefunction to be restricted to the maximum spin feasible.

Once we have this equation we can introduce an interaction; we chose the Dirac oscillator one as it is probably the simplest. This allowed us then to develop the full formalism of a relativistic particle with arbitrary spin for a particular type of interaction.

As we have no physical example of a relativistic particle with *arbitrary* spin in a Dirac oscillator potential, we shall not give graphs of the energy levels for the cases $s=1$, $s=\frac{3}{2}$, as functions of ω, N, j, that can be derived from the secular equations associated with the matrices in (4.21) and in the appendix. It is interesting, however, to note the differences between a non-relativistic oscillator with spin s and a relativistic one. In the former case all that happens is that the degeneracy of the levels increases by a factor $(2s+1)$, whereas in the latter the complexity of the spectra increases spectacularly with spin, as can be seen when we compare the secular equations derived from the matrices in (4.7), (4.10), with those in (4.21), (4.29) and in the appendix, associated with $s = \frac{1}{2}, 1, \frac{3}{2}$, respectively.

We plan to extend the analysis presented here to other interactions of relativistic particles with higher spin and, in particular, to those in which comparison with experiment is feasible.

Appendix

This is a 16×16 numerical matrix corresponding to $n = 3$, whose determinants gives the secular equation from which we can obtain the energy E as a function of the number of quanta N, the total angular momentum j and the frequency ω. We first write the matrix in terms of four 8×8 matrices as

$$\begin{bmatrix} \mathbf{D}^{+} - 3E\mathbf{I} & \boldsymbol{\Delta} \\ \tilde{\boldsymbol{\Delta}} & \mathbf{D}^{-} - 3E\mathbf{I} \end{bmatrix}$$

Table 1(a)

where

$$\mathbf{D}^{+} = \begin{bmatrix} 3 &&&&&&& \\ & 3 &&&&&& \\ && -1 &&&&& \\ &&& -1 &&&& \\ &&&& -1 &&& \\ &&&&& -1 && \\ &&&&&& -1 & \\ &&&&&&& -1 \end{bmatrix}$$

Table 1(b)

$$\mathbf{D}^{-} = \begin{bmatrix} 1 &&&&&&& \\ & 1 &&&&&& \\ && 1 &&&&& \\ &&& 1 &&&& \\ &&&& 1 &&& \\ &&&&& 1 && \\ &&&&&& -3 & \\ &&&&&&& -3 \end{bmatrix}$$

Table 1(c)

while $\tilde{\boldsymbol{\Delta}}$ is the transposition of the 8×8 matrix $\boldsymbol{\Delta}$ whose value in terms of the coefficients (4.47)–(4.50) is given below:

$$\boldsymbol{\Delta} = \begin{bmatrix} 0 & c & c & 0 & d & 0 & 0 & 0 \\ 0 & e & \frac{2}{3}e & 0 & g & h & 0 & 0 \\ 0 & \frac{-1}{\sqrt{3}}c & \frac{-1}{\sqrt{3}}c & 0 & \frac{2}{\sqrt{3}}d & 0 & d & 0 \\ 0 & \frac{-1}{\sqrt{3}}e & \frac{-2}{3\sqrt{3}}e & 0 & \frac{2}{\sqrt{3}}g & \frac{2}{\sqrt{3}}h & g & h \\ 0 & \frac{2}{3}q & q & 0 & \frac{1}{3\sqrt{2}}u & \frac{-1}{\sqrt{6}}y & \frac{-1}{\sqrt{6}}u & \frac{1}{\sqrt{2}}y \\ \frac{2}{3}q & 0 & 0 & \frac{-1}{3}q & 0 & 0 & 0 & 0 \\ \frac{-1}{3}q & 0 & 0 & \frac{2}{3}q & 0 & 0 & 0 & 0 \\ 0 & q & \frac{2}{3}q & 0 & \frac{1}{3\sqrt{2}}u & \frac{-1}{\sqrt{6}}y & \frac{-1}{\sqrt{6}}u & \frac{1}{\sqrt{2}}y \end{bmatrix}.$$

Table 1(d)

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